WEAKLY NONLINEAR ANALYSIS OF THERMOACOUSTIC OSCILLATIONS

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In this theoretical study, a weakly nonlinear analysis is performed on a generic thermoacoustic system. The velocity and heat release are assumed to be periodic. The heat release fluctuations are characterized by their phase, ϕ, relative to the velocity fluctuations, and by their amplitude, a. Both ϕ and a are functions of the velocity amplitude, r. Around $r = 0$, ϕ must be known up to the second derivative with respect to r, and a up to the third derivative. A standard linear analysis shows that the point of linear instability (the Hopf bifurcation point) is determined only by ζ, the first derivative of a, and the zeroth derivative of ϕ (i.e. the value of ϕ when $r = 0$). The weakly nonlinear analysis shows that the type of bifurcation (supercritical or subcritical) is determined by a simple expression containing the first, second, and third derivatives of a, and the zeroth, first, and second derivatives of ϕ. The functions $\phi(r)$ and $a(r)$, which characterize a flame's response to forcing, can be measured from experiments or numerical simulations. They are called the Flame Describing Function. This analysis quickly reveals the type of bifurcation that this flame will cause and whether this behaviour is due to phase-dependence, amplitude-dependence, or some combination of the two.

1. Introduction

When Yuri Gagarin was launched into orbit in 1961 on a Vostok 1, the probability of a rocket blowing up on take-off was around 50% [1]. In those days, one of the most persistent causes of failure was a violent oscillation caused by the coupling between acoustics and heat release in the combustion chamber. These thermoacoustic oscillations have caused countless rocket engine and gas turbine failures since the 1930s and have been studied extensively [2] (§1.1–1.2). Nevertheless, they are still one of the major problems facing rocket and gas turbine manufacturers today [3].

Rockets, jet engines and power generating gas turbines are particularly susceptible to coupling between heat release and acoustics because they have high energy densities and low acoustic damping. The energy densities are roughly 10 GW m^{-3} for liquid rockets, 1 GW m^{-3} for solid rockets, and 0.1 GW m^{-3} for jet engines and afterburners [2] The acoustic damping is low because combustion chambers tend to be nearly closed systems whose walls reflect sound. Consequently, high amplitude acoustic oscillations are sustained even when a small proportion of the available thermal energy is converted to acoustic (mechanical) energy. Furthermore, because so much thermal energy is available, the existence and amplitude of thermoacoustic oscillations tend to be very sensitive to small changes in the system and therefore difficult to predict.

The simplest and most useful starting point for the study of thermoacoustic oscillations is a linear stability analysis of the steady base flow. This either considers the behaviour of perturbations that are periodic in time, or the response to an impulse. In both cases, the system is said to be linearly

stable if every small perturbation decays in time and linearly unstable if at least one perturbation grows in time. These analyses have been performed on all types of rocket and gas turbine engines [2] (§6) [4] [5] [6] Most of the work in the last 50 years has been in the framework of linear analyses.

If a combustor is linearly unstable, the amplitude of infinitesimal thermoacoustic oscillations starts to grow exponentially. This cannot continue indefinitely, however, and eventually nonlinear effects act to limit it. In the simplest cases, the system reaches a constant amplitude periodic solution. In other cases it can reach multi-periodic, quasi-periodic or chaotic solutions [7] [8].

The operating point at which the combustor transitions from linear stability to linear instability is called a bifurcation point. If the system's behaviour around this point is periodic, it is called a Hopf bifurcation. The nonlinear behaviour around this point is particularly important. On the one hand, if the growth rate decreases as the oscillations' amplitude increases, then the steady state amplitude grows gradually as the operating point passes through the Hopf bifurcation. This is known as a supercritical bifurcation. On the other hand, if the growth rate increases as the oscillations' amplitude increases, then the steady state amplitude grows increasingly as the operating point passes through the Hopf bifurcation point, until a higher order nonlinearity acts to limit it. This is known as a subcritical bifurcation.

When the position of the Hopf bifurcation has been determined with a linear analysis, it is useful to know whether this bifurcation is supercritical or subcritical. In thermoacoustic systems, supercritical bifurcations are relatively benign but subcritical bifurcations are dangerous. When the bifurcation is subcritical, not only do oscillations grow suddenly as the system passes into the linearly unstable regime, but also there are linearly stable operating points that can trigger suddenly to sustained oscillations. In other words, large amplitude thermoacoustic oscillations can start without warning.

The heat release fluctuations inside thermoacoustic systems are often characterized by their phase, relative to the acoustic velocity fluctuations, and by their amplitude. Both the amplitude of heat release and the phase of heat release are functions of the acoustic velocity amplitude. Given knowledge of the amplitude dependences of the heat release amplitude and heat release phase, the weakly nonlinear analysis presented here determines whether a Hopf bifurcation is supercritical or subcritical and reveals the relative influences of the two dependences.

2. The thermoacoustic system

The thermoacoustic system examined in this paper is a tube of length L_0 in which a velocity-coupled compact heat source is placed distance $\tilde{x}_f$ from one end [9, 10]. A base flow is imposed through the tube with velocity u_0. The physical properties of the gas in the tube are described by c_v, γ, R and λ_t, which represent the constant volume specific heat capacity, the ratio of specific heats, the gas constant and the thermal conductivity respectively. The unperturbed quantities of the base flow are ρ_0, p_0 and T_0, which represent density, pressure and temperature respectively. From these one can derive the speed of sound $c_0 \equiv \sqrt{\gamma R T_0}$ and the Mach number of the flow $M \equiv u_0/c_0$.

Acoustic perturbations are considered on top of this base flow. In dimensional form, the perturbation velocity and perturbation pressure are represented by $\tilde{u}$ and $\tilde{p}$. Quantities evaluated at the flame's position, $\tilde{x}_f$, have subscript f. The rate of heat transfer to the gas there is given by $\tilde{Q}$, which depends on u in a way that will be defined later. Acoustic damping is represented by ζ.

The dimensional momentum and energy equations for the acoustic perturbations are:

$$\rho_0 \frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{x}} = 0, \tag{1}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} + \gamma p_0 \frac{\partial \tilde{u}}{\partial \tilde{x}} + \zeta \frac{c_0}{L_0} \tilde{p} - (\gamma - 1)\tilde{\dot{Q}}\tilde{\delta}(\tilde{x} - \tilde{x}_f) = 0. \tag{2}$$

Reference scales for speed, pressure, length and time are u_0, $p_0 \gamma M$, L_0 and L_0/c_0. The dimensional variables, coordinates and Dirac delta can then be written as:

$$\tilde{u} = u_0 u, \quad \tilde{p} = p_0 \gamma M p, \quad \tilde{x} = L_0 x \quad \tilde{t} = (L_0/c_0)t, \quad \tilde{\delta}(\tilde{x} - \tilde{x}_f) = \delta(x - x_f)/L_0, \qquad (3)$$

where the quantities without a tilde or subscript 0 are dimensionless. Substituting (3) into the dimensional governing equations (1) and (2) and making use of the definition of c_0 and the ideal gas law, $p_0 = \rho_0 R T_0$, gives the dimensionless governing equations for acoustic perturbations:

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \qquad (4)$$

$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \frac{\gamma - 1}{\gamma p_0 u_0} \tilde{Q} \delta(x - x_f) = 0, \qquad (5)$$

The pipe has open ends at $x = 0$ and $x = 1$, at which $p = 0$ and $\partial u/\partial x = 0$. Only the first acoustic mode will be considered here, for which $u(x,t) = \eta \cos(\pi x)$ and $p(x,t) = -(\dot{\eta}/\pi) \sin(\pi x)$. When the higher modes are included, the analysis becomes much more complicated (but not impossible). The same is true of FDF analyses, for which the standard procedure is to consider just the first mode or to assume that the amplitudes of the higher modes are fixed multiplies of that of the first mode [11]. As a side-effect, this makes the system less non-normal [12]. The first mode is substituted into (4–5), which are then rearranged to give:

$$\ddot{\eta} + \pi^2 \eta + \zeta \dot{\eta} + q = 0. \qquad (6)$$

where $q \equiv 2\pi \tilde{Q}(\gamma - 1)/(\gamma p_0 u_0) \sin \pi x_f$. Equation (6) is similar to equation (46) in Culick [14] but with a simpler damping term and a general heat release term. The acoustic energy of this system is $E = (\eta^2 + \dot{\eta}^2)/2$ and the rate of change of energy is $\dot{E} = -\zeta \dot{\eta}^2 - q \dot{\eta}$.

In this paper, it is assumed that harmonic velocity oscillations exist of the form $\eta = r \cos(\omega t)$. These give rise to periodic (but not harmonic) oscillations in q. In a moment, the rate of change of energy will be integrated over a cycle. Only the component of q at frequency ω integrates to a non-zero value over a cycle, so this is the only component that needs to be considered. It is of the form $q = a \cos(\omega t + \phi)$. In this expression, a and ϕ are functions of r but do not vary significantly over a cycle. This assumption is equivalent to the two-timing assumption in Strogatz [13].

3. Weakly nonlinear analysis

The change in energy over one cycle, ΔE, is found by integrating $\dot{E}$ over one cycle. This gives $\Delta E = -\pi r^2 \omega \zeta - \pi r a \sin \phi$. A periodic solution exists when $\Delta E = 0$, which occurs when:

$$-a \sin \phi = \omega \zeta r \qquad (7)$$

The weakly nonlinear analysis is performed around the Hopf bifurcation point, where periodic solutions have small amplitude: $r \ll 1$. This means that MacLaurin expansions of a and ϕ are valid:

$$a = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots \qquad (8)$$
$$\phi = \phi_0 + \phi_1 r + \phi_2 r^2 + \ldots \qquad (9)$$

where $a_1 = \mathrm{d}a/\mathrm{d}r|_{r=0}$ etc. . Note that $a_0 = 0$ because q is the heat release perturbation about the steady state. The zeroth order phase, ϕ_0, is not zero.

In equation (7), $\sin \phi$ is required at small r. The phase, ϕ, is not small, however, so the small angle approximations cannot be used. The expression is simplified by noting that ϕ can be written as $\phi = \phi_0 + \psi(r)$, where $\psi(r)$ is a small number. Therefore:

$$\sin \phi = \sin(\phi_0 + \psi) = \cos \phi_0 \sin \psi + \sin \phi_0 \cos \psi \qquad (10)$$

and now the small angle approximations can be used for $\sin \psi$ and $\cos \psi$. To second order in r, this gives:

$$\sin \phi \approx \cos \phi_0 \left(\phi_1 r + \phi_2 r^2 \right) + \sin \phi_0 \left(1 - \frac{1}{2}(\phi_1^2 r^2) \right) \tag{11}$$

Grouping these terms in increasing powers of r gives:

$$\sin \phi \approx \sin \phi_0 + r \left[\phi_1 \cos \phi_0 \right] + r^2 \left[\phi_2 \cos \phi_0 - \frac{1}{2}\phi_1^2 \sin \phi_0 \right] \tag{12}$$

Substituting these into (7) gives a cubic expression for the value of r at a periodic solution:

$$r \left[a_1 \sin \phi_0 \right] + r^2 \left[a_2 \sin \phi_0 + a_1(\phi_1 \cos \phi_0) \right] \ldots$$
$$\ldots + r^3 \left[a_3 \sin \phi_0 + a_2(\phi_1 \cos \phi_0) + a_1(\phi_2 \cos \phi_0 - \frac{1}{2}\phi_1^2 \sin \phi_0) \right] = -\omega\zeta r \tag{13}$$

There is a trivial solution with $r = 0$. It can be shown that the nature of the bifurcation is determined by the terms in r and r^3. For now we will consider the special case when the term in r^2 is zero. In this special case, equation (13) reduces to

$$- \left[a_1 \sin \phi_0 \right] - r^2 \left[a_3 \sin \phi_0 + a_2(\phi_1 \cos \phi_0) + a_1(\phi_2 \cos \phi_0 - \frac{1}{2}\phi_1^2 \sin \phi_0) \right] = \omega\zeta \tag{14}$$

This can be written as:

$$-r^2 \left[\frac{a_3}{a_1} + \frac{a_2\phi_1 + a_1\phi_2}{a_1 \tan \phi_0} - \frac{1}{2}\phi_1^2 \right] = \frac{\omega\zeta}{a_1 \sin \phi_0} + 1 \tag{15}$$

The position of the Hopf bifurcation is found by setting $r = 0$, which gives:

$$\omega\zeta = -a_1 \sin \phi_0 \tag{16}$$

The type of bifurcation is determined by the sign of the term in square brackets on the left hand side of (15). For example, when ζ is positive and $\sin(\phi_0)$ is negative, which is the physically realistic case, the bifurcation is supercritical when the term in square brackets is negative and subcritical when it is positive.

As the amplitude, r, of the velocity perturbation increases, the amplitude, a, and phase, ϕ, of the heat release changes nonlinearly. One often wonders which of these has more influence: the amplitude-dependence of the heat release amplitude, or the amplitude-dependence of the heat release phase. As regards the type of bifurcation, this can now be quantified by finding the Flame Describing Function and then examining the relative influences of the terms in the expression:

$$\left[\frac{a_3}{a_1} + \frac{a_2\phi_1 + a_1\phi_2}{a_1 \tan \phi_0} - \frac{1}{2}\phi_1^2 \right] \tag{17}$$

The first of these depends only on the heat release amplitude and is the only remaining term in the case where the phase is constant. Its effect has been noted before [15]. The third term depends quite sensitively on changes in the the heat release phase. The second term is a combination of the two.

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