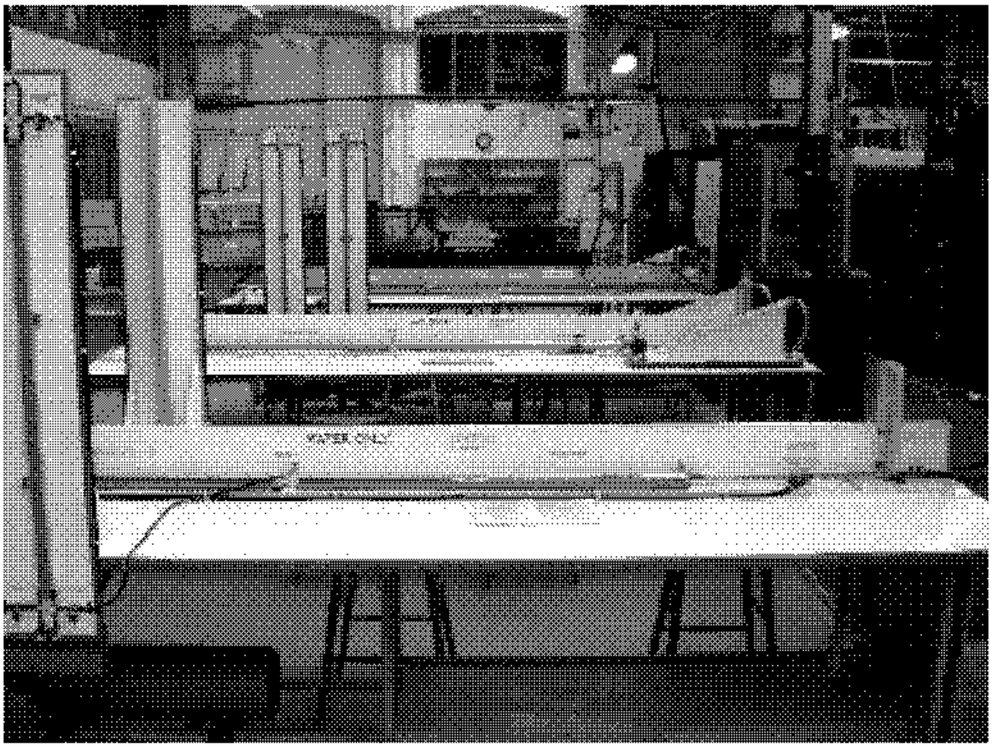
Chapter 5

The pipe flow experiment

- Definition of the friction coefficient
- Laminar flow in a circular pipe
- Turbulent flow in a circular pipe
- Mixing, momentum transport and eddy viscosity
- Roughness

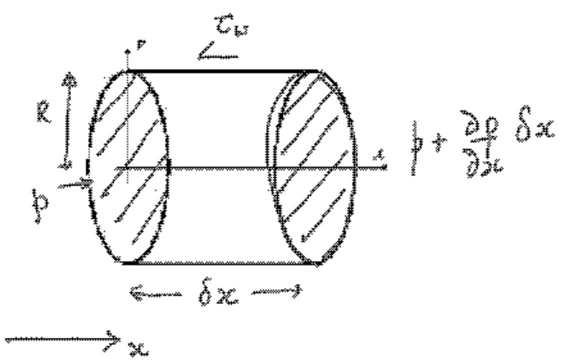


5.1 Definition of the friction coefficient

The friction coefficient, c_f, is the non-dimensional wall shear stress, τ_w. The wall shear stress is a force per unit area and has dimensions Nm^{-2}. It can be made non-dimensional by the dynamic pressure: $\rho V^2/2$, where V is the average velocity in the pipe, defined as the volumetric flowrate divided by the cross-sectional area.

$$\text{friction coefficient} = c_f = \frac{\text{wall shear stress}}{\text{dynamic pressure}} = \frac{\tau_w}{\frac{1}{2}\rho V^2}$$

The pipe flow in the experiment is allowed to develop fully before reaching the test section. This means that the velocity profile is uniform in the x-direction and therefore that there is no change in momentum along the control volume shown below. Therefore the pressure drop along the pipe, $\mathrm{d}p/\mathrm{d}x$, must balance the wall shear stress:



$$-2\pi R\,\delta x\,\tau_w - \pi R^2 \frac{\partial p}{\partial x}\delta x = 0$$

p is a function of x only

$$\Rightarrow \tau_w = -\frac{\pi R^2}{2\pi R}\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\Rightarrow \tau_w = -\frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}x}$$

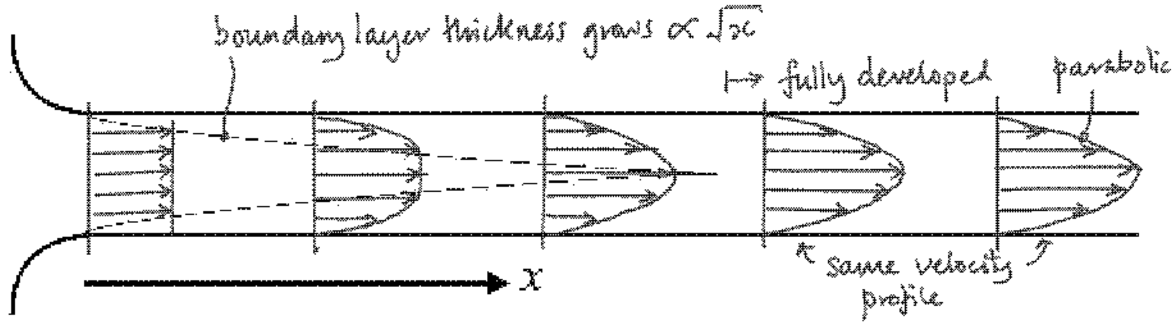
Therefore, for fully-developed pipe flow (whether laminar or turbulent) the friction coefficient can be expressed in terms of the pressure drop:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho V^2} = -\frac{R}{\rho V^2}\frac{\mathrm{d}p}{\mathrm{d}x}$$

minus sign depends on the sign convention for τ_w

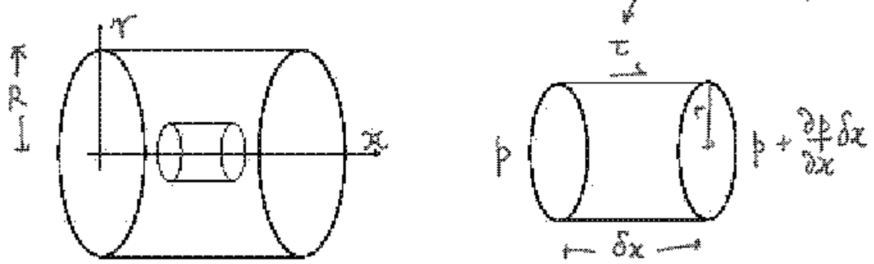
5.2 Laminar flow in a circular pipe

The friction coefficient can be calculated exactly when a viscous fluid is forced slowly down a pipe. The boundary layers grow with $\sqrt{x}$ and eventually meet. At this point the flow is fully developed and has the familiar Poiseuille flow (parabolic) profile.



The worked example on pages 9 and 10 of chapter 3 shows one way to derive the velocity profile. An easier way to derive the velocity profile is to balance the forces on cylindrical elements centred on the centreline:

define as positive in the x-direction



$$2\pi r \delta x \tau - \pi r^2 \frac{\partial p}{\partial x} \delta x = 0$$

$$\tau = \frac{r}{2}\frac{dp}{dx}$$

$$\tau = \mu \frac{dv_x}{dr}$$

$$\mu \frac{dv_x}{dr} = \frac{r}{2}\frac{dp}{dx}$$

This gives the velocity gradient in terms of the pressure gradient. We need to work out the velocity profile by integrating the expression and applying the no slip boundary condition:

$$\int_0^{v_x} dv_x = \int_R^{r} \frac{r'}{2\mu}\frac{dp}{dx}\, dr'$$

$$v_x = \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{r'^2}{4\mu}\right]_R^r$$

parabolic

$$v_x = -\frac{\mathrm{d}p}{\mathrm{d}x}\left(\frac{R^2 - r^2}{4\mu}\right)$$

always positive

velocity is always in the opposite direction to the pressure gradient

Now we need to evaluate the average velocity in the pipe, V, in terms of the pressure drop. This is the total flowrate divided by the cross-sectional area:

$$
\begin{aligned}
\text{average velocity} = V &= \overbrace{\frac{1}{\pi R^2}}^{\text{cross-section area}} \overbrace{\int_0^R v_x(r) 2\pi r dr}^{\text{total flowrate}} = \frac{1}{\pi R^2}\int_0^R -\frac{\mathrm{d}p}{\mathrm{d}x}\left(\frac{R^2 - r^2}{4\mu}\right) 2\pi r dr \\
\Rightarrow V &= -\frac{1}{2\mu R^2}\frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R \\
\Rightarrow V &= -\frac{R^2}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \quad \leftarrow \text{average velocity} \propto \text{axial pressure gradient} \\
\Rightarrow \frac{dp}{dx} &= -\frac{8\mu}{R^2} V
\end{aligned}
$$

We have already determined an expression for the friction coefficient, c_f, in terms of the pressure drop. Now we can express it in terms of the average flowrate by substituting in the above expression:

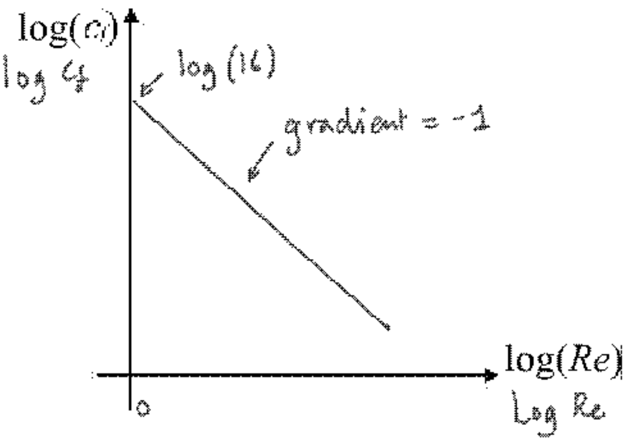
$$
\begin{aligned}
c_f &= -\frac{R}{\rho V^2}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{R}{\rho V^2}\frac{8\mu V}{R^2} \\
c_f &= \frac{8\mu}{\rho V R}
\end{aligned}
$$

average velocity in the pipe

$$
\text{Define } Re \equiv \frac{\rho V D}{\mu} = 2\frac{\rho V R}{\mu}
$$

$$
c_f = \frac{16}{Re}
$$

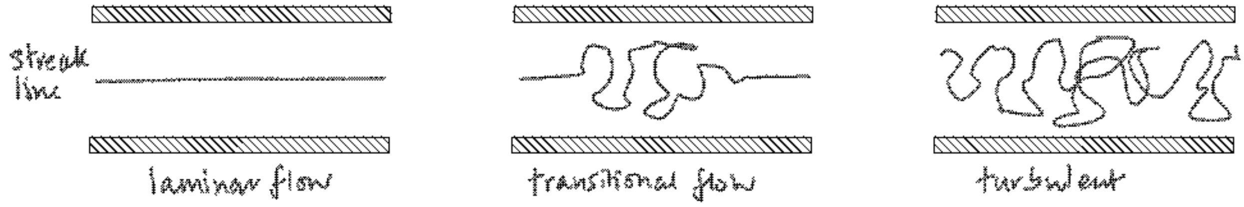
So, for a laminar pipe flow, $c_f = 16/Re$; i.e. the friction coefficient only depends on the Reynolds number. The results from the experiment match this theoretical result very well in the laminar flow region.



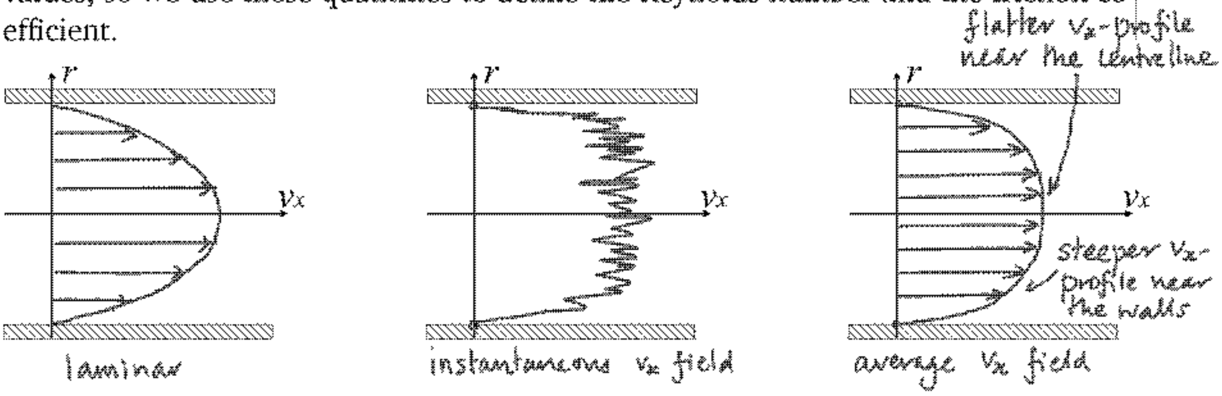
$$
\begin{aligned}
C_f &= 16/Re \\
\log C_f &= \log(16/Re) \\
\log C_f &= \log 16 - \log Re
\end{aligned}
$$

5.3 LAMINAR FLOW, TURBULENT FLOW, AND MIXING

If the flow were to remain laminar, the friction coefficient would always decrease with increasing Reynolds number. However, above a certain Reynolds number, the friction coefficient increases to a new value, which is independent of Reynolds number. This indicates that the assumptions in our model have broken down. A close inspection of the fluid in the pipe reveals that little perturbations in the flow are growing rapidly and the flow has become turbulent.



When turbulent, the flow is highly unsteady, meaning that the velocity and pressure at a point in space vary with time. The *time-averaged* quantities, however, have steady values, so we use these quantities to define the Reynolds number and the friction coefficient.

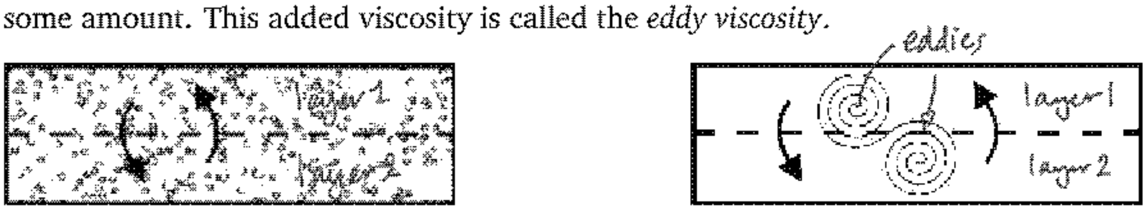


In a laminar flow, molecular diffusion is the only transport process between layers of fluid. Laminar flows are therefore very hard to mix. They need to be *folded* rather than *stirred*. In a turbulent flow, packets of fluid move between layers of fluid in turbulent eddies. This occurs on a much larger scale than molecular diffusion so the mixing rate is much larger.

Consequently, the rate of momentum transport in a turbulent flow is much greater than that in a laminar flow. There is a higher rate of momentum transfer from the fluid to the pipe walls, therefore a higher shear stress and a greater pressure drop.

5.4 Mixing, momentum transport and eddy viscosity

In chapter 3 we considered the transport of momentum due to molecular diffusion between layers of fluid and modelled this with a viscosity coefficient. In a turbulent flow, whole eddies move between layers, greatly increasing the rate of transport of momentum. We can *model* the effect of turbulence by increasing the viscosity by some amount. This added viscosity is called the *eddy viscosity*.



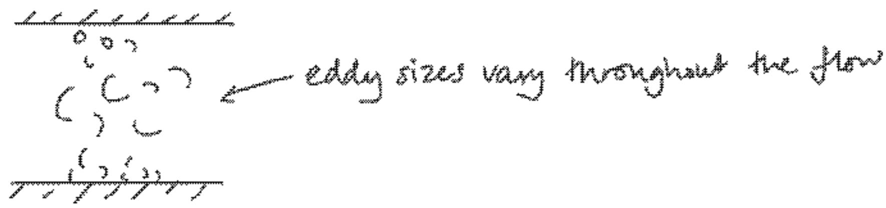
laminar flow: molecular mixing

$$\tau = \mu \frac{\partial v_x}{\partial y} \qquad \mu \ll \mu_T \qquad \tau_T = \mu_T \frac{\partial v_x}{\partial y}$$

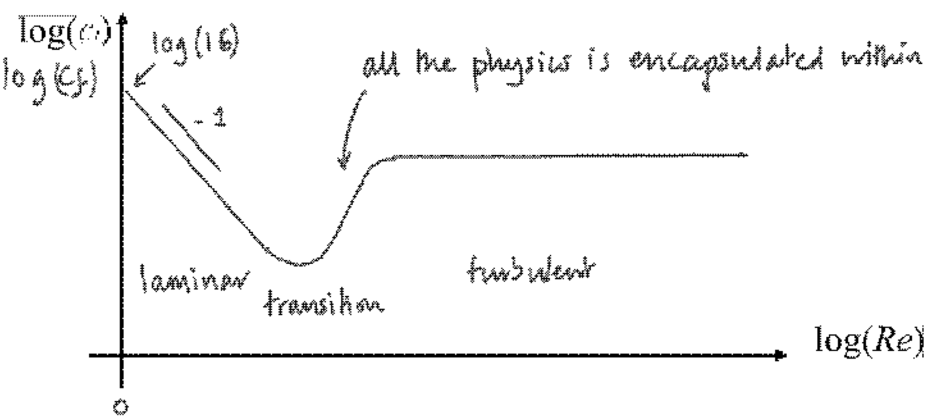
turbulent flow: eddies mix

eddy viscosity

The eddy viscosity depends on the eddy size. Eddies come in all sizes and are larger at the centre and smaller at the walls. Their sizes and shapes are strongly dependent on the flow upstream, for example whether it has just gone round a bend. Consequently, the eddy viscosity varies throughout the fluid in ways that we find difficult to model. This is why the concept of eddy viscosity, although useful for envisaging the flow, is not always successful in calculations. In fact, finding a universal description of turbulence is often described as the last unsolved problem in classical mechanics.

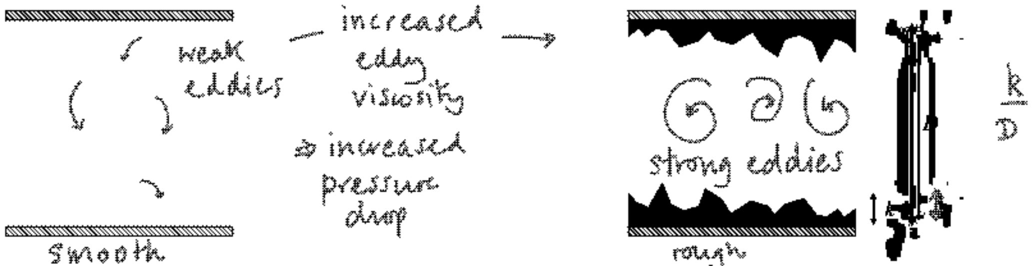


Nevertheless, in a smooth straight pipe, the friction coefficient *only* depends on the Reynolds number. We measure this dependence experimentally, knowing that it is valid for *all* smooth straight pipes. We need never know exactly what is going on within the pipe because all the physics is encapsulated within the plot of $c_f(Re)$.

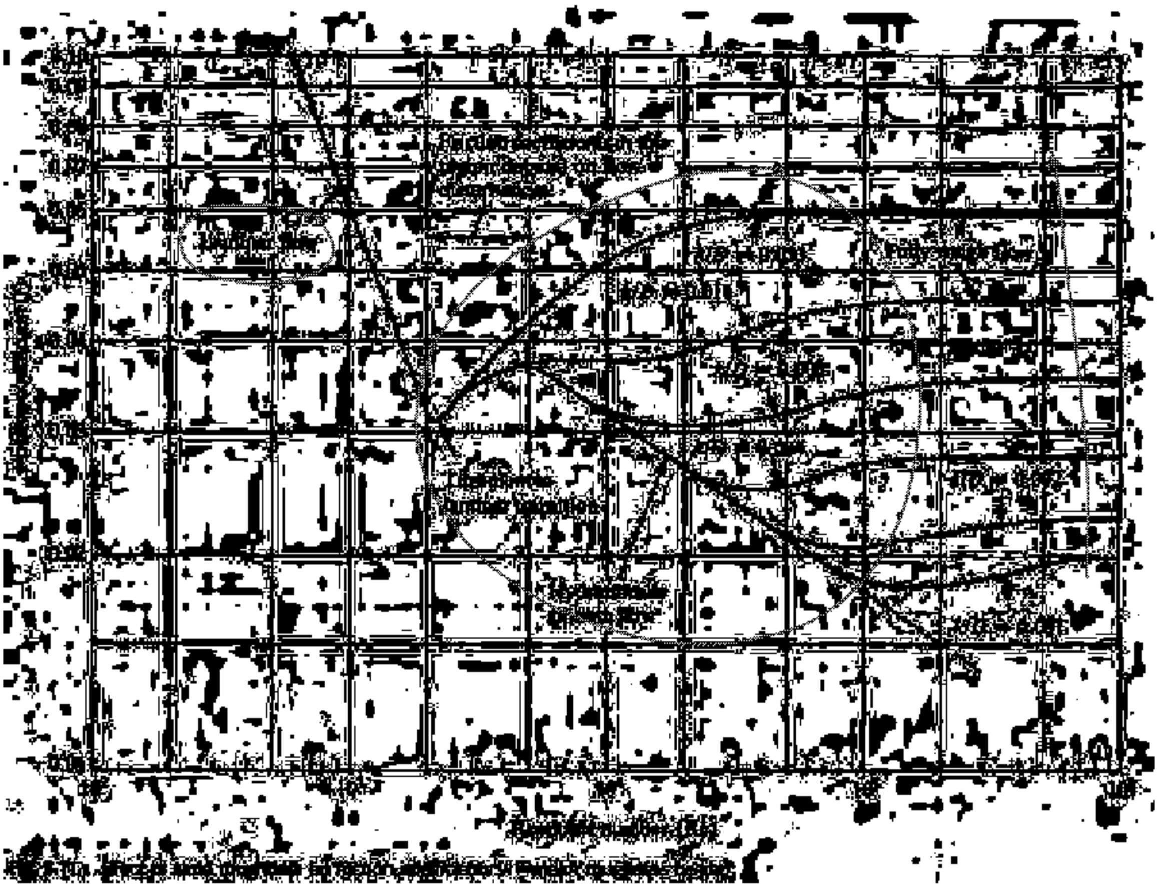


5.5 Roughness

The roughness of a pipe is defined as the ratio of the average bump size to the diameter of the pipe. The roughness affects the sizes of turbulent eddies in the flow, which then affect the eddy viscosity. In general, rough pipes create large eddies which increase the eddy viscosity and therefore increase the pressure drop along the pipe.



Again, it is impossible to predict exactly what is going on within the pipe and we rely on empirical formulations (e.g. experiments such as the pipe flow experiment). These give the friction coefficient as a function of roughness and Reynolds number and are called Moody charts. All the physics is encapsulated within the lines of $c_f(Re; k/D)$.



From *Internal Flow Systems* by D. S. Miller,
ISBN 0-947711-77-5 (TA379 & TA293)