

# CHAPTER 1

## FLUIDS AND VECTOR CALCULUS

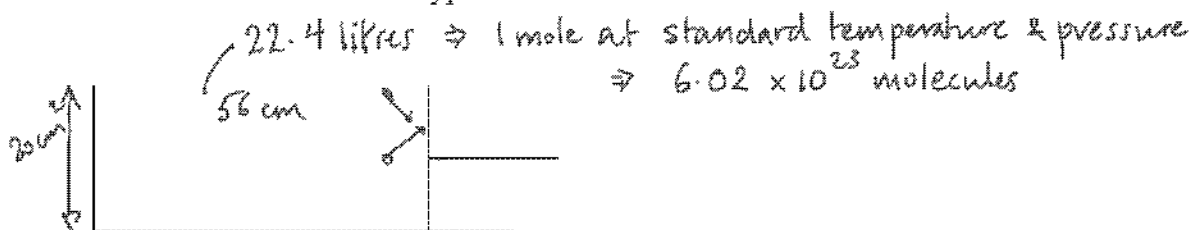
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- Picturing fluids
- The properties of fluids
- Why we use partial derivatives
- The del operator
- The gradient of a scalar field
- The law of conservation of mass
- The divergence of a vector field
- Constant density flows
- The curl of a vector field

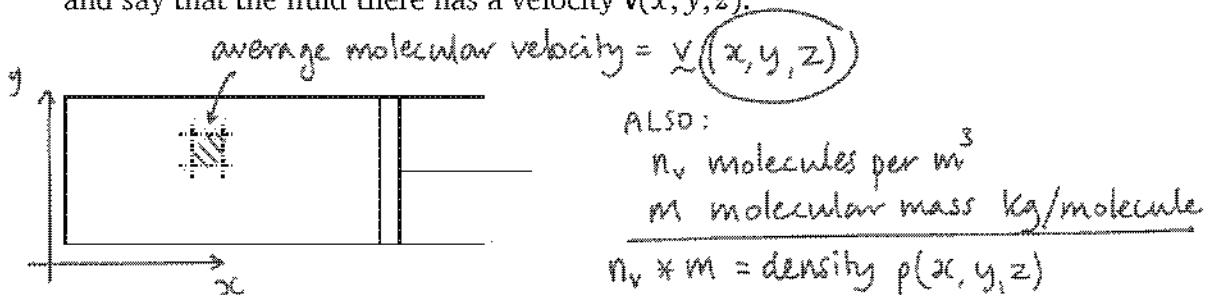
## 1.1 PICTURING FLUIDS

Fluids flow. This is because they cannot support a shear stress when in static equilibrium.<sup>1</sup> Fluids are made up of very many molecules<sup>2</sup> and we can picture these molecules as small rigid balls that obey Newton's laws of motion. In gases, the molecules move in straight lines between one collision and the next with a *mean free path* that is much larger than the molecular diameter. In liquids, the molecules are in close contact with their neighbours, colliding and forming temporary bonds. These temporary bonds make some liquids difficult to model so we will consider gases first.

Given that gas molecules obey Newton's laws, could we simulate the behaviour of a gas by tracking the position of each molecule? In theory, we could. But how many molecules are there in a typical container?



At all but the smallest lengthscales it would be impractical to follow every molecule. Instead, we average<sup>3</sup> all the molecular velocities,  $\mathbf{v}_i$ , around a point in space  $(x, y, z)$  and say that the fluid there has a velocity  $\mathbf{v}(x, y, z)$ .



So now we can think of the fluid as a *continuum*, which is a continuous lump of stuff *with no gaps* and say that it has a certain velocity *field*.

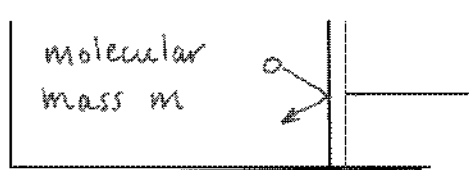
<sup>1</sup>To define 'static equilibrium' exactly we need to define a timescale over which we observe the material. This is important because some materials, such as ice in a glacier, are solid over short timescales but liquid over long timescales.

<sup>2</sup>This statement implies that we observe fluids at lengthscales very much larger than the distance between the molecules.

<sup>3</sup>This requires a definition of *average*. This is  $\mathbf{v}(x, y, z) = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i$ , where  $N$  is the number of molecules around that point in space and  $\mathbf{v}_i$  is the velocity vector of each molecule. The volume over which we average must be much larger than the mean free path but much smaller than the observer's lengthscale.

## 1.2 THE PROPERTIES OF FLUIDS

The properties of a fluid all arise from its molecular nature. The molecular mass,  $m$ , multiplied by the number of molecules in one metre cubed,  $n_v$ , gives the density,  $\rho$ . The temperature,  $T$ , is proportional to the average kinetic energy of the molecules,  $\overline{mv_i^2}/2$ . For example, if we heat up a stationary gas, the *speeds* of all the molecules increase although, of course, their *velocity vectors* still average to zero because the gas remains stationary. To show how useful this concept is, let's look at the pressure on the piston face<sup>4</sup>:



① momentum change on collision  $\propto m v_i$   <sup>$i=1 \rightarrow v$</sup>

② number of collisions per unit area per unit time  $\propto n_v v_i$  <sub>"average"</sub>

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{\text{average rate of change of momentum}}{\text{area}} = \textcircled{1} \times \textcircled{2}$$

$$= \text{constant} \times n_v \overline{m v_i^2} = \text{constant} \times n_v \times \overline{m v_i^2}$$

$$= n_v m \left( \frac{2 \times \text{constant}}{m} \right) \frac{\overline{m v_i^2}}{2}$$

density
gas constant
temperature

$$p = \rho R T$$

So by considering the molecular motion of the gas we can work out that  $p = \rho R T$ , where  $R$  is the gas constant. This is the *ideal gas equation of state*. In a similar way, macroscopic properties like viscosity, thermal conductivity, specific heat capacity etc. can be worked out from the microscopic molecular motion.

The Knudsen number,  $Kn$ , is defined as the ratio of  $\lambda$ , the mean free path, to  $L$ , the size of the region we are considering. The continuum model only works when we average over very many molecules and very many collisions so it requires the Knudsen number to be somewhat<sup>5</sup> less than 1. Consequently the continuum model does not work at very low pressures or for very small objects.

<sup>4</sup>For a more rigorous derivation of this, see Feynman's lectures on physics, section 39.

<sup>5</sup>For now, we will estimate breakdown to occur at  $Kn = 1$ , although in some situations it could break down at smaller  $Kn$ .

### 1.3 WHY WE USE PARTIAL DERIVATIVES

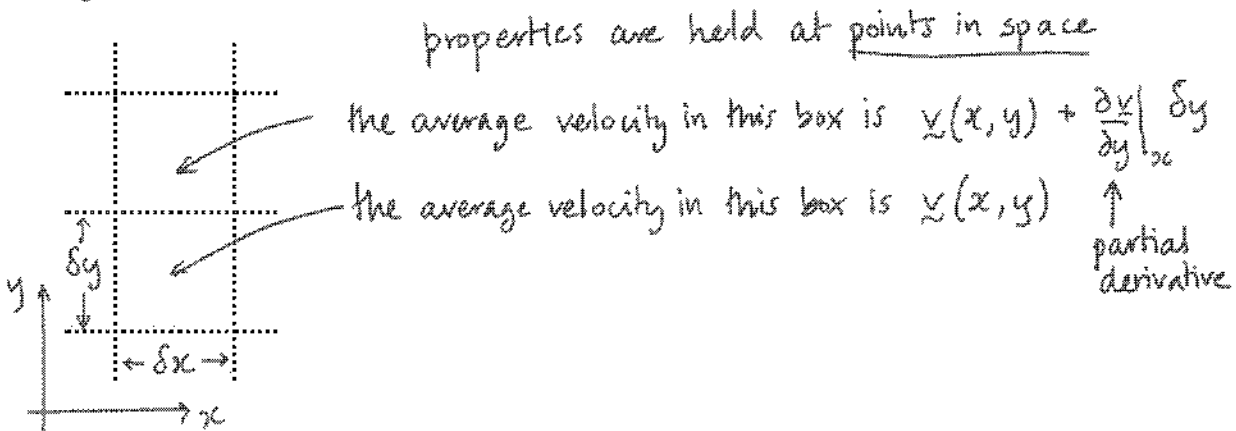
In our molecular picture of a fluid, every molecule has a velocity vector,  $v_i$ , and obeys Newton's laws of motion. The velocity is held *by the molecule* so we use *ordinary* derivatives such as  $d/dt$ . If we knew exactly how all the molecules started we could march forwards in time solving ordinary differential equations for each molecule. However, as mentioned in section 1.1, this is impractical for more than a few million molecules.



$v_i$  is held by the molecule  
 so we use ordinary derivatives

$$v_i = \frac{dx_i}{dt}$$

In our continuum picture of a fluid, the velocity field is defined at *points in space* rather than on individual molecules. Newton's laws of motion still work but we have to apply them to regions of space rather than to individual molecules. In a three-dimensional world we need to isolate changes in each of the three directions:  $(x, y, z)$  so we need to use *partial* derivatives such as  $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ . By averaging over very many molecules and very many collisions we exchange an enormous number of *ordinary* differential equations for a few *partial* differential equations that describe the averaged fields.



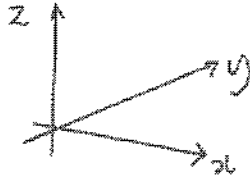
This is a crucial conceptual leap for physicists and engineers.

*"The application of Newton's mechanics to continuously distributed masses led inevitably to the discovery and application of partial differential equations, which in their turn first provided the language for the laws of the field-theory."*

ALBERT EINSTEIN

### 1.4 THE DEL OPERATOR

When dealing with point masses, we use *ordinary* differential operators such as  $d/dt$ . When dealing with fields we need to use *partial* differential operators such as  $\partial/\partial t$ ,  $\partial/\partial x$ ,  $\partial/\partial y$  and  $\partial/\partial z$ . Partial differential operators are not very useful when they act independently; they only generate the change in one direction and, even worse, that direction depends on the choice of coordinate system.



$\frac{\partial}{\partial x} |_{y,z}$  is not very useful by itself

The real power of partial differential operators arises when they are combined to form the *del* operator, which is given the symbol  $\nabla$  and is also called *nabla*. In Cartesian coordinates,  $\nabla$  is defined as:

$$\nabla \equiv \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

↑  
del

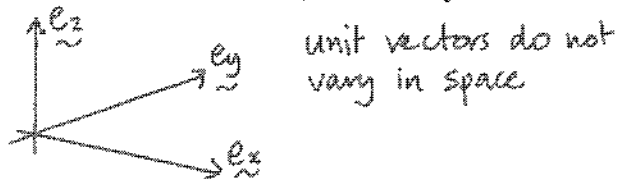
unit vector in the x-direction

The operation that is represented by  $\nabla$  is *independent* of the coordinate system. This means that  $\nabla$  is expressed differently in different coordinate systems. For instance, in cylindrical polars it is:

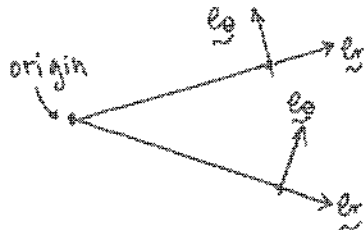
$$\nabla \equiv \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

; no convenient shorthand

In a Cartesian coordinate system, the unit vectors are the same everywhere. When  $\nabla$  acts on another vector, we do not need to worry about the effect that  $\nabla$  has on the unit vectors because  $\partial \hat{e}_x / \partial x$ ,  $\partial \hat{e}_y / \partial x$  etc. are all zero. (This is why the Cartesian shorthand works for  $\nabla$ ).



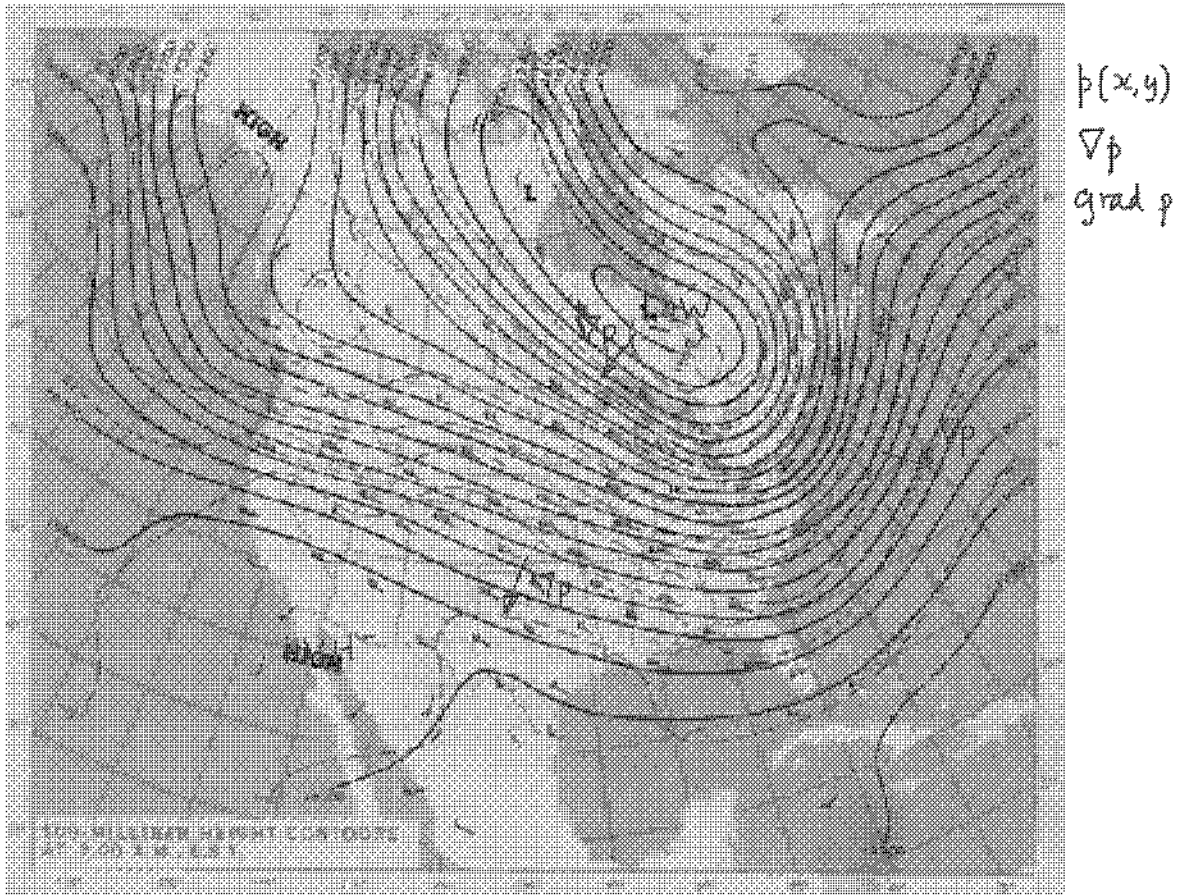
In other coordinate systems, the unit vectors are not the same everywhere. This means that, when  $\nabla$  acts on a vector, its effect on the unit vectors must also be taken into account using the product rule. We encounter this in chapter 2.



$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta ; \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

## 1.5 THE GRADIENT OF A SCALAR FIELD

When  $\nabla$  acts directly on a scalar field, it produces a vector that points in the direction of steepest increase of that scalar. Its magnitude equals the *gradient* in that direction. For example this weather map shows lines of constant pressure above the USA.



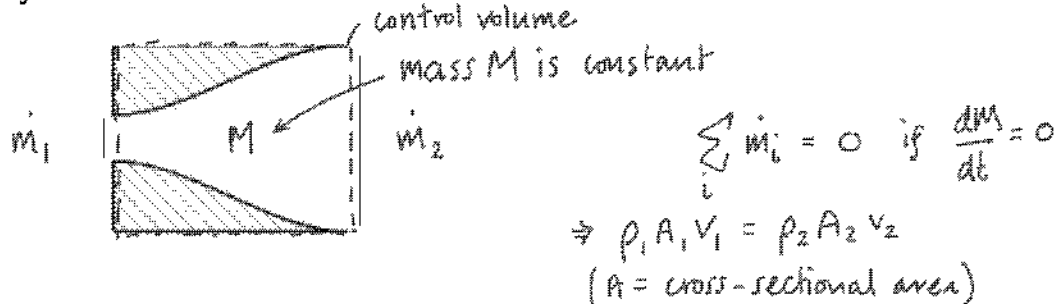
$\nabla p$  is known as “grad  $p$ ” because it gives the gradient of  $p$  at all points in the field.

$$\nabla p = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} p = \begin{bmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{bmatrix} \text{ a vector}$$

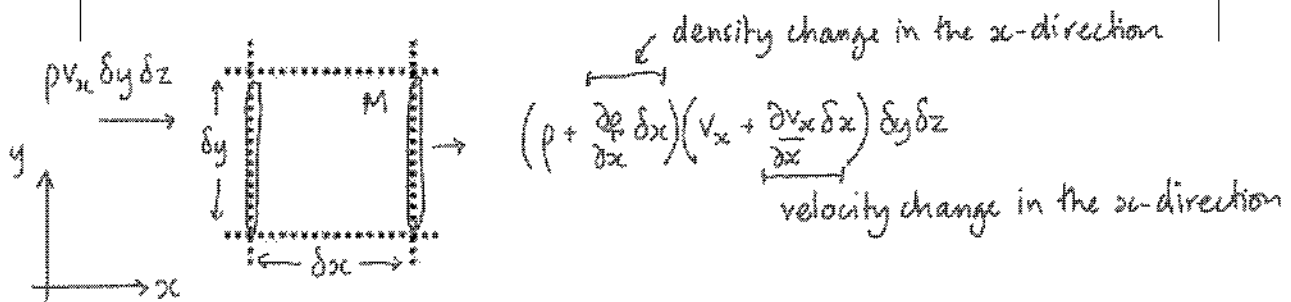
The vector  $\nabla p$  is orthogonal to the contour lines and points in the direction in which the pressure increases.

## 1.6 THE LAW OF CONSERVATION OF MASS

You may be familiar with the law of conservation of mass in one dimension:



$$\frac{\dot{m}_1}{A_1} = \frac{\rho_1 A_1 v_1}{A_1} = \rho_1 v_1 \quad \text{units: } \frac{\text{kg}}{\text{m}^2 \text{ s}}$$



In a time  $\delta t$ , the change in mass within the volume,  $\delta M$ , is given by

$$\delta M = \left\{ \begin{array}{l} \text{in} \quad \quad \quad \text{out} \quad \quad \quad \text{in} \quad \quad \quad \text{out} \\ \rho v_x \delta y \delta z - \left( \rho + \frac{\partial \rho}{\partial x} \delta x \right) \left( v_x + \frac{\partial v_x}{\partial x} \delta x \right) \delta y \delta z + \rho v_y \delta x \delta z - \left( \rho + \frac{\partial \rho}{\partial y} \delta y \right) \left( v_y + \frac{\partial v_y}{\partial y} \delta y \right) \delta x \delta z \end{array} \right\} \delta t$$

$$\Rightarrow \frac{\delta M}{\delta t} = - \left( \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} \right) \delta x \delta y \delta z$$

But  $M = \rho \delta x \delta y \delta z$  so the  $\delta x \delta y \delta z$  cancels and, as  $\delta t$  tends to zero, we obtain:

$$\frac{\partial \rho}{\partial t} = - \left( \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} \right) = - \left[ \begin{array}{c} \partial/\partial x \\ \partial/\partial y \end{array} \right] \cdot \left[ \begin{array}{c} \rho v_x \\ \rho v_y \end{array} \right] = - \nabla \cdot (\rho \underline{v})$$

"divergence"  
"del dot mass flux"  
"divergence of mass flux"

This is the law of conservation of mass: the rate of change of mass per unit volume (density) is the net rate at which mass flows out of the volume. This method of deriving the formula is easy to understand physically but requires some messy maths. There is an equivalent derivation using Gauss' theorem.

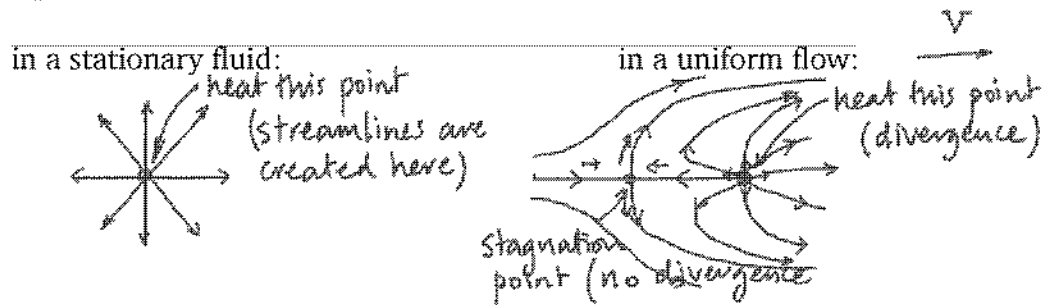
### 1.7 THE DIVERGENCE OF A VECTOR FIELD

When  $\nabla$  is dotted with a vector field  $\mathbf{a}$ , it produces a scalar field equal to the net flux of  $\mathbf{a}$  out of each point in space. This is known as the *divergence* of the field  $\mathbf{a}$ .

For example, we saw in section 1.6 that the quantity  $\rho\mathbf{v}$  is a vector field that represents the mass flux. We know from the law of conservation of mass that:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v}) = - \text{divergence of the mass flux}$$

If we heat up a fluid at a point in space, perhaps with a laser, its density drops at that point – i.e.  $\partial\rho/\partial t$  is negative – and the fluid *diverges* away from that point – i.e.  $\nabla \cdot (\rho\mathbf{v})$  is positive.



- 1. mass *diverges* from heated spot.
- 1. streamlines *created* at heated spot.
- 2. no net divergence from the stagnation point.

### 1.8 THE EQUATION OF STATE AND CONSTANT DENSITY FLOWS

The equation of state of a fluid gives the relationship between the density,  $\rho$ , the pressure,  $p$ , and the temperature,  $T$ . There is no universal equation of state that accurately models the properties of all fluids under all conditions. Instead, various different models are used. The most familiar model is that for an ideal gas ( $\rho = p/(RT)$ ). A more sophisticated model is the Van der Waals equation of state, which accounts for the finite volume occupied by the molecules themselves. You can calculate  $\rho(p, T)$  with both models but, obviously, it makes no sense to use both densities simultaneously. You have to choose one model and reject all the others.

*These cannot be used together.*

In the rest of this Fluid Mechanics course, we will use the simplest possible equation of state:  $\rho = \text{constant}$ . In other words, the density is neither a function of pressure nor of temperature. This is the *incompressible fluid* model. In chapter 10, we will determine when this is a valid model. Calculations become much easier with the incompressible fluid model. For example:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v}) = -\rho \nabla \cdot \underline{\underline{v}} - \underline{\underline{v}} \cdot \nabla \rho$$

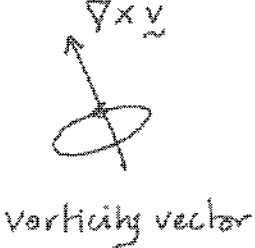
but  $\frac{\partial \rho}{\partial t} = 0$  and  $\nabla \rho = 0 \Rightarrow \nabla \cdot \underline{\underline{v}} = 0$

*in an incompressible fluid, streamlines cannot be created or destroyed*



## 1.9 THE CURL OF A VECTOR FIELD

Finally,  $\nabla \times \mathbf{a}$  is known as *curl a*. When  $\nabla$  is crossed with the velocity field,  $\mathbf{v}$ , it produces a vector whose magnitude is twice the angular velocity of the fluid at each point in space and whose orientation is the axis of rotation at that point. This is called the *vorticity*.

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}$$


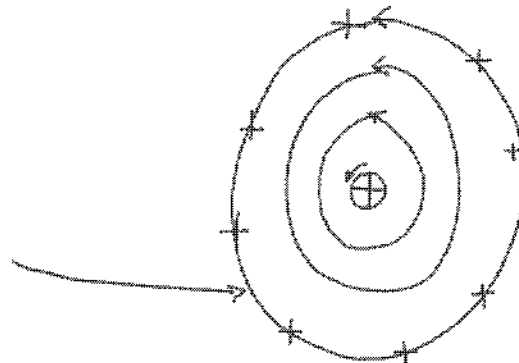
It is important to note that fluids can move along curved paths without actually rotating. For instance, in a plughole vortex, only the fluid near the centre has non-zero vorticity. The fluid blobs away from the centre move along circular paths but keep facing in the same direction. You can test this by putting a matchstick cross onto the surface:

Over the plughole:

- the fluid spins quickly (high shear);
- viscous forces are strong;
- the fluid has non-zero vorticity.

Away from the plughole the cross keeps pointing in the same direction

- low shear;
- viscous forces are weak;
- the fluid has zero vorticity.



You may have been told that Bernoulli's equation cannot be applied across streamlines. This is the safe approach but it isn't quite the whole story. In the next chapter we will discover that Bernoulli's equation can be applied across streamlines but only in special cases in which there is no vorticity between the streamlines.