CHAPTER 3

VISCOUS FLOW

- Solids and fluids;
- The no slip condition;
- Momentum transfer through molecular motion;
- Shear stress and viscosity;
- Couette flow;
- Poiseuille flow;
- The Navier-Stokes equation;
- Worked example: viscous pipe flow.

3.1   SOLIDS ARE RIGID. FLUIDS FLOW

Imagine holding a brick between the palms of your hands. When you move your right hand away from you and your left hand towards you, the forces from your hand are transmitted through the brick. This is because the molecules in the brick have defined positions. When displaced slightly, the inter-molecular bonds resist in a spring-like manner. In static equilibrium, each layer experiences the same shear stress:

Now imagine that the brick is replaced with treacle. When you move your hands, the treacle flows into a new shape. This is because the molecules do not have defined positions and cannot support a shear stress when in static equilibrium. When one layer is displaced, they flow over each other to accommodate the displacement. There *is* a shear stress, however, in the moments before mechanical equilibrium is reached. We will soon see that it is proportional to the rate of strain, dv_x/dy.

What would happen if we replaced the treacle with the inviscid fluid that we have considered so far? An inviscid fluid has perfect slip. It cannot support any shear stress at all. It would be the perfect lubricant (if you could keep it in the desired place).[1]

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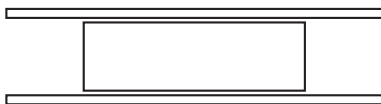
[1]Inviscid fluids do actually exist. They are called *superfluids*. The best known example is liquid helium-4 below 2.17 Kelvin. At these temperatures quantum mechanics needs to be taken into account and the classical mechanics models used in this course break down.

3.2   THE NO SLIP CONDITION AND MOMENTUM TRANSFER

Let us consider the molecules in a gas between two plates, where the top plate moves from left to right. From experiments we find that the molecules stick to surfaces for long enough to reach thermal equilibrium before they jump back into the gas. Consequently when they leave the surface they have, on average, the same x-velocity and the same temperature as the surface. This is the *no slip condition*.
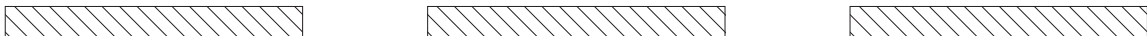
The molecules that have just left the surface collide into molecules nearby. After several collisions the extra x-momentum of the molecules coming from the top surface has been diffused into adjacent layers of fluid. These in turn jostle with the molecules adjacent to them, transferring x-momentum deeper into the fluid. Eventually x-momentum diffuses right down to the bottom plate and, averaging over all the molecules' velocities, one obtains a linear velocity profile. This is just like diffusion of heat, in which one obtains a linear temperature profile.
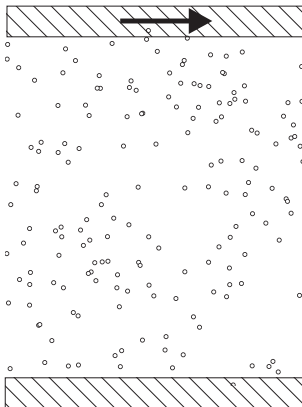
3.3 SHEAR STRESS AND VISCOSITY

There are too many molecules to consider each individually so we consider the fluid as a continuum. The transfer of momentum by molecular motion then needs to be modelled in some way. Adjacent layers of fluid exchange momentum at a rate that is proportional to the velocity gradient. By Newton's second law ($\mathbf{f} = m\mathbf{a}$) the rate of change of momentum across a certain area is simply a force. When divided by the area, this is the shear stress τ:
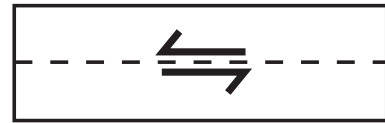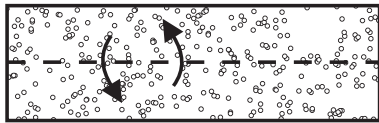
In a certain time and over a certain area, some molecules swap places

$$\text{shear stress} = \frac{\text{shear force}}{\text{area}}$$

The coefficient of proportionality is the viscosity, μ: (see footnote[2])

$$\tau = \mu \frac{d v_x}{d y}$$

Viscosity varies strongly with temperature because it is closely linked to molecular motion. In gases, viscosity increases with temperature because the average molecular speed increases and the momentum transfer per unit time therefore increases. However in liquids it decreases with increasing temperature, as you can see when you pour boiling water out of a kettle. This reflects the fact that the molecules in a liquid do not simply bounce off each other. Instead they form temporary bonds with each other which enhance the transfer of momentum. This bond energy becomes less significant compared with their kinetic energy as the temperature increases.

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[2]In this particular case v_x is *only* a function of y so the partial derivative $\partial v_x / \partial y$ collapses to the ordinary derivative $d v_x / d y$.

For most fluids, the rate at which momentum diffuses is proportional to the velocity gradient, just as the rate at which heat diffuses is proportional to the temperature gradient. These fluids are called '*Newtonian fluids*'. If the molecules are long chains or the fluid contains small suspended solids, these can align or distort with the flow direction so the viscosity depends on the velocity gradient. These fluids are called '*non-Newtonian fluids*'.
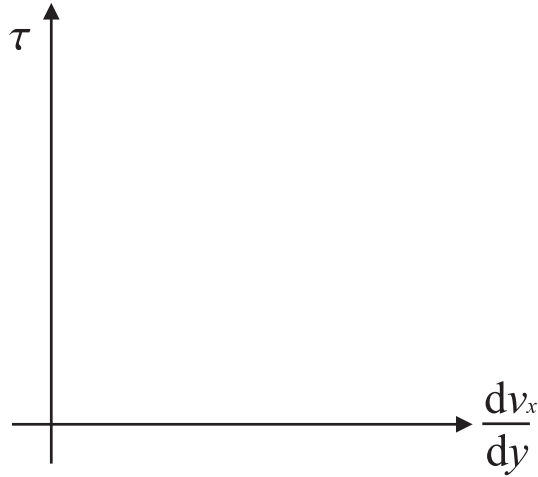
3.4   COUETTE FLOW

The steady incompressible viscous flow between moving plates described in the previous section is called 'Couette flow'. There is no acceleration (see bottom of page) so, by $\mathbf{f} = m\mathbf{a}$, the forces on a control volume must sum to zero:
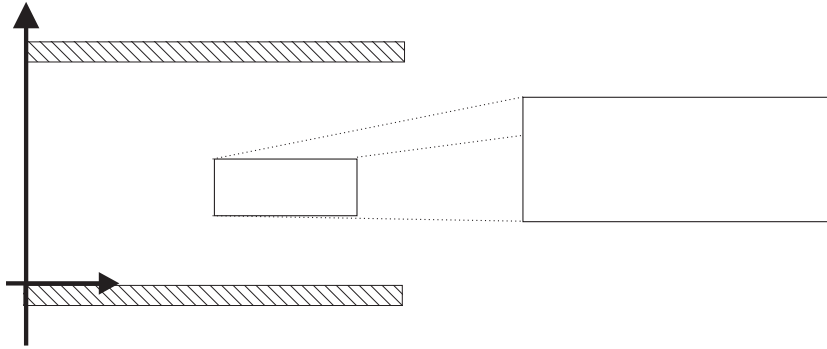
In Couette flow, the pressure gradient is zero so:

From our molecular argument we know that we can model the shear stress in terms of the viscosity of the fluid and the velocity gradient. Therefore we substitute $\tau = \mu d v_x / d y$ into this expression:
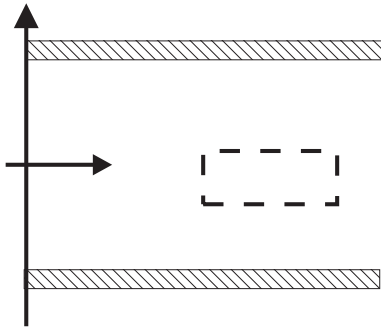
This has solution: $v_x = By + C$. The constants B and C are evaluated from the boundary conditions:

Before we started we should have looked at the *material derivative of velocity* (i.e. the acceleration of a fluid blob) to show that the flow is not accelerating:

$$\frac{d\mathbf{v}}{dt} = \frac{D\mathbf{v}}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$
$$= \frac{\partial \mathbf{v}}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right)\mathbf{v}$$

3.5   POISEUILLE FLOW

Now we consider the flow between stationary plates with a pressure gradient. The velocity profile is obtained from a force balance on the same control volume:

$$\frac{d\tau}{dy}\delta y \delta x - \frac{dp}{dx}\delta x \delta y = 0$$
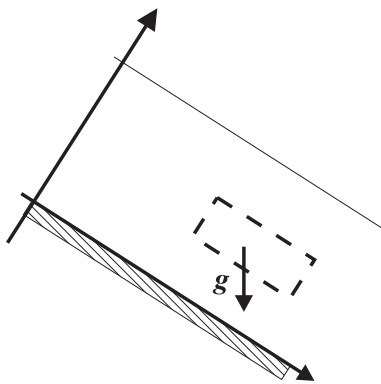
$$\Rightarrow \quad \frac{d\tau}{dy} = \frac{dp}{dx}$$

$$\Rightarrow$$

$$\Rightarrow \quad v_x = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)y^2 + By + C$$

$$\Rightarrow \quad v_x = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)\left(y^2 - \frac{D^2}{4}\right) \tag{3.1}$$

3.6   VISCOUS FLOW DOWN A SLOPE

A similar force balance can be performed on a steady viscous flow down a slope.

3.7   THE NAVIER-STOKES EQUATION

The Navier-Stokes equation is the most important equation in fluid mechanics. It is simply $\mathbf{f} = m\mathbf{a}$ for a viscous fluid.

In chapter 2 we derived the Euler equation, which is $\mathbf{f} = m\mathbf{a}$ for an inviscid fluid:

$$-\nabla p = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

When the shear stresses are included, there are extra force terms on the left hand side. These are shown here for a Newtonian fluid such as water or air:

$$\mu \frac{\partial^2 \mathbf{v}}{\partial x^2} + \mu \frac{\partial^2 \mathbf{v}}{\partial y^2} + \mu \frac{\partial^2 \mathbf{v}}{\partial z^2} - \nabla p = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

In vector notation, this can be written as:

$$\mu (\nabla \cdot \nabla) \mathbf{v} - \nabla p = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

which is abbreviated to:

$$\mu \nabla^2 \mathbf{v} - \nabla p = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

This is the Navier-Stokes equation. It is $\mathbf{f} = m\mathbf{a}$ written for a viscous fluid. It is usually written the other way round and divided by ρ :

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \mathbf{v}$$

3.8 WORKED EXAMPLE - VISCOUS PIPE FLOW

We would like to work out the velocity profile $v_x(r)$ in a pipe of radius R with a pressure gradient dp/dx. The control volume is a cylindrical shell of thickness δr and length δx. Note that the outer surface of this shell has a larger area than the inner surface. We assume that v_x and τ only vary in the r-direction and that p only varies in the x-direction, which means that our partial derivatives collapse to ordinary derivatives.
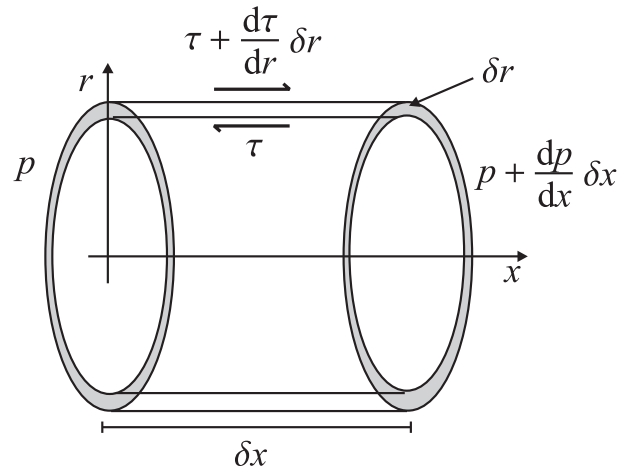
If the flow is steady and R is constant, the forces on the fluid element must sum to zero:

$$(2\pi r \delta r)p - (2\pi r \delta r)\left(p + \frac{dp}{dx}\delta x\right) + 2\pi(r + \delta r)\left(\tau + \frac{d\tau}{dr}\delta r\right)\delta x - (2\pi r \tau)\delta x = 0$$

Multiplying out the brackets, cancelling terms and dropping the very small $(\delta r)^2$ term gives:

$$-(2\pi r \delta r)\frac{dp}{dx}\delta x + 2\pi r \frac{d\tau}{dr}\delta r \delta x + 2\pi \delta r \tau \delta x = 0$$

which reduces to:

$$-\frac{dp}{dx} + \frac{d\tau}{dr} + \frac{\tau}{r} = 0$$

Now we substitute in $\tau = \mu dv_x/dr$:

$$-\frac{dp}{dx} + \mu \frac{d^2 v_x}{dr^2} + \frac{\mu}{r}\frac{dv_x}{dr} = 0$$

As for the 2D case, we need to integrate this twice in order to find $v_x(r)$. However, this is difficult when there are three terms. The trick is to notice that the second two terms can be re-written as a single term:

$$-\frac{dp}{dx} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right) = 0$$

Now we integrate this, leaving in constants (A and B) instead of specifying the bounds of the integration:

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right) = \frac{dp}{dx}$$

$$\Rightarrow \quad r\frac{dv_x}{dr} = \int \frac{dp}{dx}\frac{r}{\mu}dr = \frac{dp}{dx}\frac{r^2}{2\mu} + B$$

$$\Rightarrow \quad \frac{dv_x}{dr} = \frac{dp}{dx}\frac{r}{2\mu} + \frac{B}{r} \tag{3.2}$$

$$\Rightarrow \quad \int dv_x = \int \frac{dp}{dx}\frac{r}{2\mu}dr + \int \frac{B}{r}dr$$

$$\Rightarrow \quad v_x = \frac{dp}{dx}\frac{r^2}{4\mu} + B\ln r + A$$

This is the general solution for v_x. We now need to put in the boundary conditions. The first is that the velocity gradient dv_x/dr must tend to zero as r tends to zero. From inspection of equation (3.2), this requires $B = 0$. The second is that $v_x(R) = 0$, which is the no slip condition at the pipe walls. The constant A is calculated from this to give the solution:

$$v_x = -\frac{dp}{dx}\left(\frac{r^2 - R^2}{4\mu}\right)$$

and we recognise the parabolic profile that is familiar from Poiseuille flow between two plates.