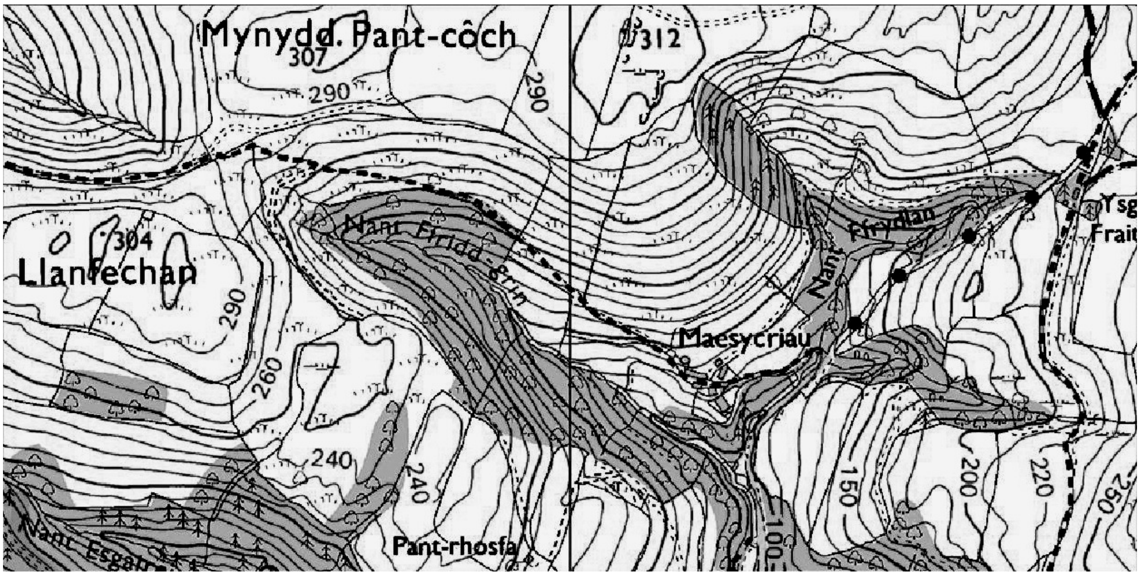
Chapter 2

Inviscid flow

- Changes due to motion through a field;
- Newton's second law ($\mathbf{f} = m\mathbf{a}$) applied to a fluid: Euler's equation;
- Euler's equation integrated along a streamline: Bernoulli's equation;
- Bernoulli's equation and streamline curvature;
- Determining the pressure field from a flow's streamlines.

2.1 Changes due to motion through a field

The contour lines on this map of Snowdonia show the height above sea level.



At every point (x, y), this two-dimensional scalar *height field* has a single value of $h(x, y)$. As you walk along a path at a certain velocity $\mathbf{v}$ at what rate does your height change?

The rate of change of your height is given by $\mathbf{v}\cdot\nabla h$. We can write this as $\mathbf{v}\cdot(\nabla h)$ or as $(\mathbf{v}\cdot\nabla)h$. These are equivalent but the second version becomes more convenient later because $(\mathbf{v}\cdot\nabla)$ is a scalar operator that acts on anything to its right. In this case it acts on h but later it will act on other variables, including vectors.

$$(\mathbf{v}\cdot\nabla) = \left(\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \right) = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right)$$

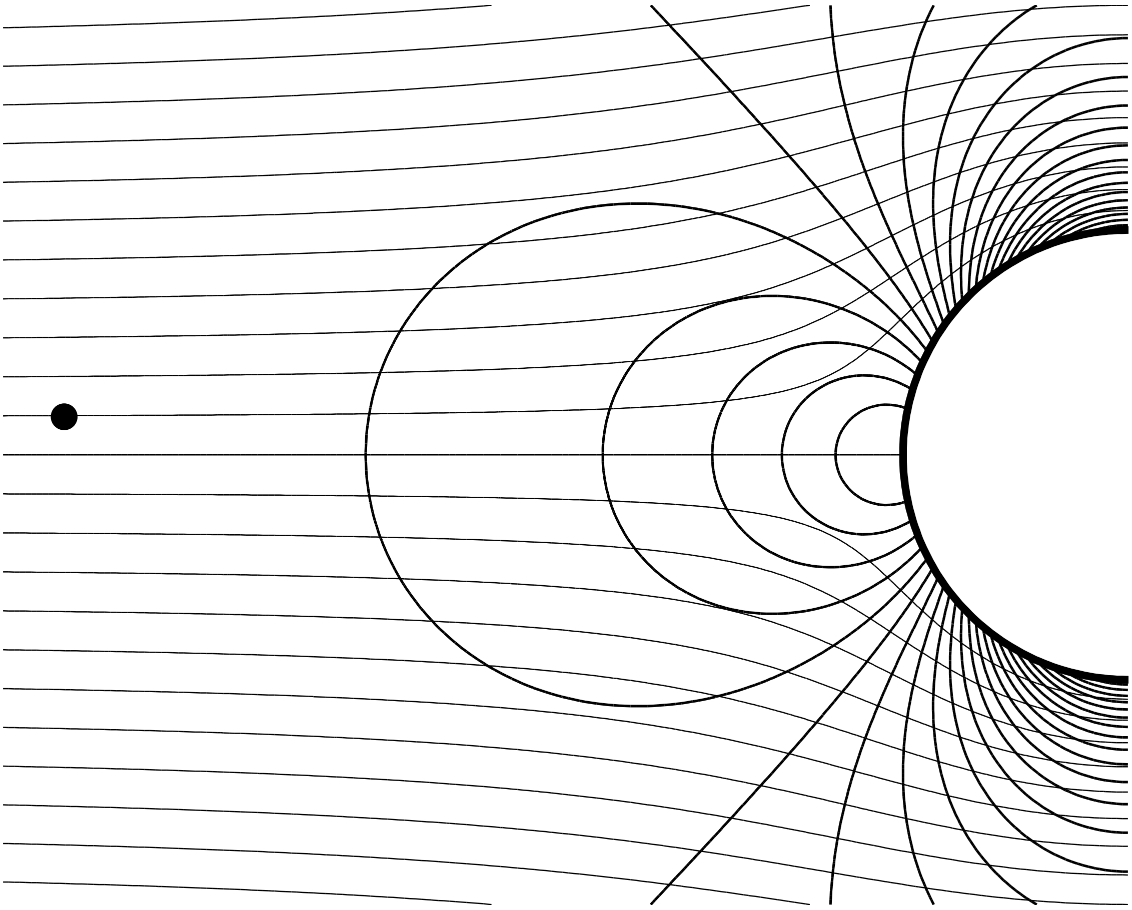
Note that $(\mathbf{v}\cdot\nabla)$ is *not* the same as $(\nabla\cdot\mathbf{v})$, which is:

$$(\nabla\cdot\mathbf{v}) = \left(\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \right) = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

On an unrelated note, it is worth mentioning that the scalar quantity v^2 is:

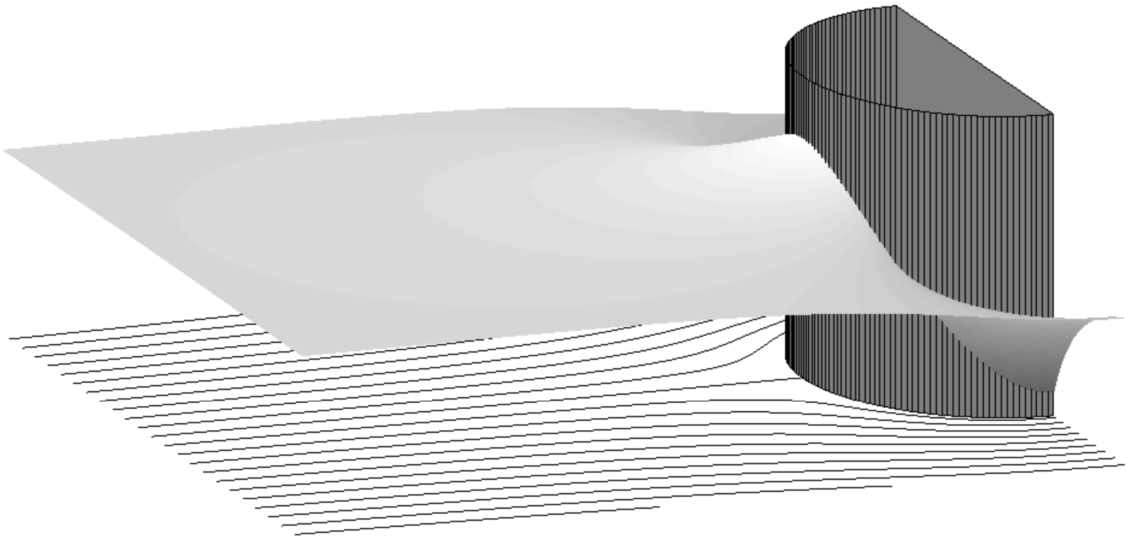
$$v^2 \equiv |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = v_x^2 + v_y^2 + v_z^2$$

The streamlines and the pressure field in the steady inviscid flow around a cylinder are shown below. We shall follow a 'blob' of fluid through this pressure field. The blob is big enough that it contains many billions of molecules, so that we can average the molecular motion and speak meaningfully about the blob's velocity and pressure. But it is small enough that we can consider the blob at a single point in the field. Being part of the fluid, the blob must move along a streamline. At what rate does its pressure change?



The rate of change of the blob's pressure is given by $(\mathbf{v} \cdot \nabla)p$.

When water flows past an obstacle, its height follows the same pattern:



Sometimes the flow will be *unsteady*, meaning that the pressure field will change with time. We simply add on the unsteady component:

$$\frac{dp}{dt} = \left.\frac{\partial p}{\partial t}\right|_{(x,y,z)} + (\mathbf{v}\cdot\nabla)p$$

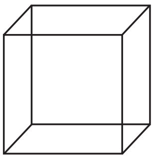
This is one of the most useful concepts in fluid mechanics. Gabriel Stokes found himself using this derivative 'following the blob' so much that he gave it a name and its own notation: the *material derivative* or *substantial derivative*

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)$$

It is coordinate-free (i.e. the definition does not depend on the coordinate system being used). Being an operator, it is hungry for something to differentiate. For instance, we can use it to find the acceleration of a fluid blob by differentiating the blob's velocity:

2.2 Newton's second law: f = ma

If we put a neutrally buoyant solid cube into a fluid flow and ignore all the viscous forces, we can work out the net force on the cube by considering the pressure on each face:



$$\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = -\begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{pmatrix} \delta x \delta y \delta z = \; -\nabla p \;\; \delta x \delta y \delta z$$

Now we write $\mathbf{f} = m\mathbf{a}$ for the cube:

We could do this for an imaginary cube (or blob) of fluid instead. Similarly, we would find that the cube (or blob) of fluid is being accelerated or decelerated by the pressure gradients in the flow. However, we know how to express the acceleration of a fluid blob in terms of the fluid's velocity field because we worked it out at the end of the previous section. We can put them together to obtain:

This is the *Euler equation*, which becomes the *Navier–Stokes equation* when viscous terms are included (see p9 of the databook, where it is called by its other name: the *momentum equation*). The Euler equation is simply $\mathbf{f} = m\mathbf{a}$ applied to an inviscid fluid. Remember that there is one equation for each dimension. For example in 3D cartesian coordinates there is one equation in the x-direction, one in the y-direction and one in the z-direction.

Link to the Momentum Equation applied to a control volume

You may have derived the Steady Flow Momentum Equation (SFME) and the more general Momentum Equation (p7 of databook) by considering a control volume:

$$\frac{d}{dt}\int_{Vol} \rho \mathbf{v} \; dV \quad + \quad \int_{Surf} \rho \mathbf{v}(\mathbf{v}\cdot \mathbf{dA}) \quad = \quad \mathbf{F} \quad - \quad \int_{Surf} p \;\; \mathbf{dA}$$

rate of change of momentum in the volume	net rate at which momentum enters the volume	body forces (e.g. gravity)	pressure forces on the surface

The Euler equation is the same momentum equation, but is applied to a point in space and is in vector notation:

$$\frac{\partial \rho \mathbf{v}}{\partial t} \quad + \quad \rho(\mathbf{v}\cdot\nabla)\mathbf{v} \quad = \quad \text{body forces} \quad - \quad \nabla p$$

2.3 Euler's equation along a straight streamline

We will look at the stagnation streamline of the steady flow around a cylinder. The fluid on this streamline flows at velocity $V(x)$ in the x-direction. This streamline is easy to examine because the streamline coordinate system is aligned with the cartesian coordinate system. The more general case is considered in the next section. The cartesian unit vectors are $(\mathbf{e_x}, \mathbf{e_y})$ and ∇ is defined as $\nabla \equiv \mathbf{e_x}\partial/\partial x + \mathbf{e_y}\partial/\partial y$.

Euler's equation (for an incompressible fluid) is:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p$$

but the y-velocity is zero and the flow is steady, so this equation becomes:

Re-arranging the $\mathbf{e_x}$ component gives:

$$\frac{\partial}{\partial x}\left(\frac{1}{2}\rho V^2\right) + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\left(p + \frac{1}{2}\rho V^2\right) = 0$$

and this can be integrated along the streamline to give:

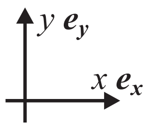
This is *Bernoulli's equation*. It has arisen naturally from $\mathbf{f} = m\mathbf{a}$ and the assumption that viscous forces can be neglected. When we integrated Euler's equation along a streamline, energy terms arose from force terms in exactly the same way that the change in potential energy of a mass in a gravitational field is equal to the force integrated over the distance it moves. Indeed, the calculation can easily be repeated with gravity, which introduces an extra $\rho g z$ term. So each term can be thought of as an *energy per unit volume*. In an inviscid flow, Bernoulli's equation applies along any streamline. *Furthermore, if there is no vorticity in the flow then the total pressure is uniform and we can apply Bernoulli's equation across streamlines.*[1]

[1]You will prove this in the examples paper.

2.4 Bernoulli and streamline curvature

We can do the same analysis for flows in which the streamlines are not aligned with the cartesian coordinate system. The great advantage of vector notation is that it applies in any coordinate system:

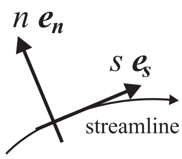
Cartesian coordinates



$$\nabla \equiv \mathbf{e_x}\frac{\partial}{\partial x} + \mathbf{e_y}\frac{\partial}{\partial y}$$

The advantage is that the unit vectors $\mathbf{e_x}$ and $\mathbf{e_y}$ do not change in space.

Instrinsic (streamline) coordinates



$$\nabla \equiv \mathbf{e_s}\frac{\partial}{\partial s} + \mathbf{e_n}\frac{\partial}{\partial n}$$

The advantage is that the velocity vector is simply $\mathbf{v} = V\mathbf{e_s}$. The disadvantage is that the unit vectors change as you go along a streamline. For example, $\partial \mathbf{e_s}/\partial s = -\mathbf{e_n}/R$, where R is the radius of curvature at that point on the streamline.

The Euler equation in steady flow can be re-arranged to:

$$\mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p$$

Applying the intrinsic definition of ∇ and noting that $\mathbf{e_s}\cdot\mathbf{e_s} = 1$ and $\mathbf{e_s}\cdot\mathbf{e_n} = 0$:

$$\begin{aligned}
V\mathbf{e_s}\cdot\left(\mathbf{e_s}\frac{\partial(V\mathbf{e_s})}{\partial s} + \mathbf{e_n}\frac{\partial(V\mathbf{e_s})}{\partial n}\right) &= -\frac{1}{\rho}\left(\mathbf{e_s}\frac{\partial p}{\partial s} + \mathbf{e_n}\frac{\partial p}{\partial n}\right) \\
V\frac{\partial(V\mathbf{e_s})}{\partial s} &= -\frac{1}{\rho}\left(\mathbf{e_s}\frac{\partial p}{\partial s} + \mathbf{e_n}\frac{\partial p}{\partial n}\right) \\
V\left(\mathbf{e_s}\frac{\partial V}{\partial s} + V\frac{\partial \mathbf{e_s}}{\partial s}\right) &= -\frac{1}{\rho}\left(\mathbf{e_s}\frac{\partial p}{\partial s} + \mathbf{e_n}\frac{\partial p}{\partial n}\right) \\
\mathbf{e_s}V\frac{\partial V}{\partial s} - V^2\frac{\mathbf{e_n}}{R} &= -\frac{1}{\rho}\left(\mathbf{e_s}\frac{\partial p}{\partial s} + \mathbf{e_n}\frac{\partial p}{\partial n}\right)
\end{aligned}$$

Resolving separately in the $\mathbf{e_s}$ and $\mathbf{e_n}$ directions gives:

$$V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \text{along a streamline (as before)}$$

$$\frac{V^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n} \qquad \text{across a streamline (new)}$$

The first of these integrates to give Bernoulli's equation, as before. The second describes how streamlines are bent by pressure gradients. Both equations are given on p9 of the databook.

2.5 Determining the pressure field from the streamlines

Now we have the tools to determine the pressure field in a flow from the streamline shape. Along streamlines we can use Bernoulli's equation. Across streamlines we need to look at the streamline curvature. For an aerofoil at a positive angle of attack: